Coherent and squeezed states of the radiation field

Following the development of the quantum theory of radiation and with the advent of the laser, the states of the field that most nearly describe a classical electromagnetic field were widely studied. In order to realize such 'classical' states, we will consider the field generated by a classical monochromatic current, and find that the quantum state thus generated has many interesting properties and deserves to be called a *coherent state*.* An important consequence of the quantization of the radiation field is the associated uncertainty relation for the conjugate field variables. It therefore appears reasonable to propose that the wave function which corresponds most closely to the classical field must have *minimum* uncertainty for all times subject to the appropriate simple harmonic potential.

In this chapter we show that a displaced simple harmonic oscillator ground state wave function satisfies this property and the wave packet oscillates sinusoidally in the oscillator potential without changing shape as shown in Fig. 2.1. This *coherent* wave packet always has minimum uncertainty, and resembles the classical field as nearly as quantum mechanics permits. The corresponding state vector is the *coherent state* $|\alpha\rangle$, which is the eigenstate of the positive frequency part of the electric field operator, or, equivalently, the eigenstate of the destruction operator of the field.

Classically an electromagnetic field consists of waves with well-defined amplitude and phase. Such is not the case when we treat the field quantum mechanically. There are fluctuations associated with both the amplitude and phase of the field. An electromagnetic field in a number state $|n\rangle$ has a well-defined amplitude but completely

* The coherent state concept was introduced by Schrödinger [1926]. For an excellent treatment of the subject see the Les Houches lectures of Glauber [1965].

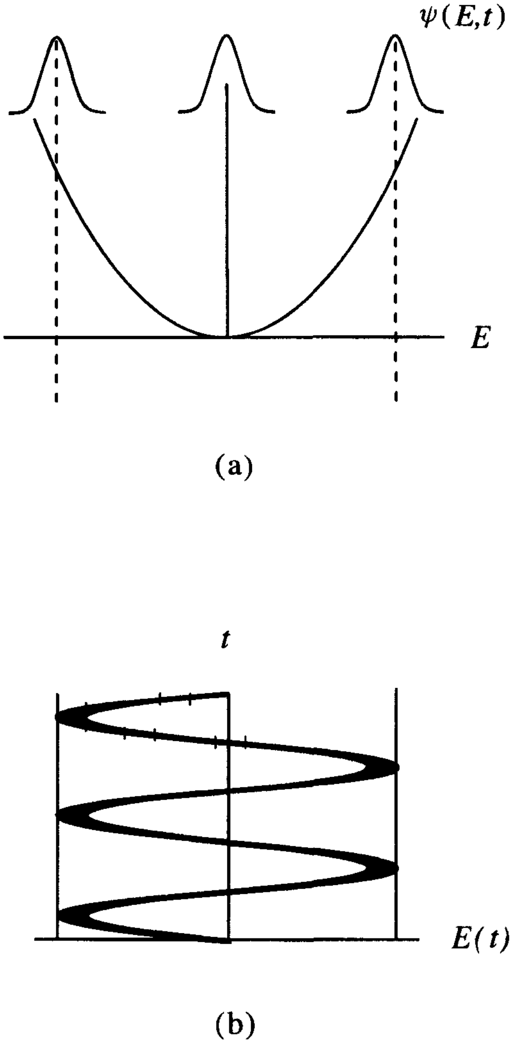


Fig. 2.1
(a) Minimum-uncertainty wave packet at different times in a harmonic oscillator potential.
(b) Corresponding electric field.

uncertain phase, whereas a field in a coherent state has equal amount of uncertainties in the two variables. Equivalently, we can describe the field in terms of the two conjugate quadrature components. The uncertainties in the two conjugate variables satisfy the Heisenberg uncertainty principle such that the product of the uncertainties in the two variables is equal to or greater than half the magnitude of the expectation value of the commutator of the variables (see Eq. (2.6.2) below). A field in a coherent state is a minimum-uncertainty state with equal uncertainties in the two quadrature components.

After developing the coherent states of the radiation field, we turn to the so-called *squeezed states*. In principle, it is possible to generate states in which fluctuations are reduced below the symmetric quantum limit in one quadrature component. This is accomplished at the expense of enhanced fluctuations in the canonically conjugate quadrature, such that the Heisenberg uncertainty principle is not violated. Such states of the radiation field are called *squeezed states*. A quadrature of electromagnetic field with reduced fluctuations below the standard quantum limit, has attractive applications in optical communication, photon detection techniques, gravitational wave detection, and noise-free amplification. In this chapter, we physically motivate and present the definition and properties of the squeezed states, with special reference to the so-called squeezed coherent states. These states result from applying the 'squeeze operator' to the coherent state.

2.1 Radiation from a classical current

In this section, we define the coherent state and show that the radiation emitted by a classical current distribution is such a state. By *classical* we mean that the current can be described by a prescribed vector $\mathbf{J}(\mathbf{r}, t)$ which is not an operator. We consider coupling of this current to the vector potential operator (cf. Eq. (1.1.27) and Section 5.1)

$$\mathbf{A}(\mathbf{r}, t) = -i\sum_{\mathbf{k}} \frac{1}{\nu_k}\hat{\epsilon}_{\mathbf{k}}\mathscr{E}_{\mathbf{k}}a_{\mathbf{k}}e^{-i\nu_k t+i\mathbf{k}\cdot\mathbf{r}} + \text{H.c.} \tag{2.1.1}$$

The Hamiltonian that describes the interaction between the field and the current is then given by

$$\mathscr{V}(t) = \int \mathbf{J}(\mathbf{r}, t)\cdot\mathbf{A}(\mathbf{r}, t)d^3r \tag{2.1.2}$$

and the state vector $|\psi(t)\rangle$ for the combined system obeys the interaction picture Schrödinger equation

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\mathscr{V}|\psi(t)\rangle. \tag{2.1.3}$$

The vector function $\mathbf{J}(\mathbf{r}, t)$ commutes with itself at different times, but the operator $\mathbf{A}(\mathbf{r}, t)$ does not. Hence the interaction energy $\mathscr{V}(t)$ does not either, and ordinarily the Schrödinger equation cannot be integrated as

$$|\psi(t)\rangle = \exp\left[-\frac{i}{\hbar}\int_0^t dt'\mathscr{V}(t')\right]|\psi(0)\rangle. \tag{2.1.4}$$

However, the various commutators introduced in obtaining the correct integration yield (2.1.4) multiplied by an overall phase factor which we discard. With (2.1.1) and (2.1.2), the exponential in (2.1.4) becomes

$$\exp\left[-\frac{i}{\hbar}\int_0^t dt'\mathscr{V}(t')\right] = \prod_{\mathbf{k}} \exp(\alpha_{\mathbf{k}} a_{\mathbf{k}}^\dagger - \alpha_{\mathbf{k}}^* a_{\mathbf{k}}), \tag{2.1.5}$$

where the complex time-dependent amplitude $\alpha_{\mathbf{k}}$ is

$$\alpha_{\mathbf{k}} = \frac{1}{\hbar\nu_k}\mathscr{E}_{\mathbf{k}} \int_0^t dt' \int d\mathbf{r}\hat{\epsilon}_{\mathbf{k}} \cdot \mathbf{J}_\nu(\mathbf{r}, t) e^{i\nu_k t' - i\mathbf{k}\cdot\mathbf{r}}. \tag{2.1.6}$$

In Eq. (2.1.6) the dipole current $\mathbf{J}_\nu(\mathbf{r}, t)$ is given the subscript ν to denote the fact that it is a monochromatic dipole oscillating at frequency $\nu = ck$. We choose the initial state $|\psi(0)\rangle$ to be the vacuum $|0\rangle$, and the state vector (2.1.4) then becomes

$$|\psi(t)\rangle = \prod_{\mathbf{k}} \exp(\alpha_{\mathbf{k}} a_{\mathbf{k}}^\dagger - \alpha_{\mathbf{k}}^* a_{\mathbf{k}})|0\rangle_{\mathbf{k}}. \tag{2.1.7}$$

This state of the radiation field is called a coherent state and is denoted as $|\{\alpha_{\mathbf{k}}\}\rangle$. It is apparent that the multi-mode coherent state in Eq. (2.1.7) can be expressed as a product of single-mode coherent states $|\alpha_{\mathbf{k}}\rangle$:

$$|\{\alpha_{\mathbf{k}}\}\rangle = \prod_{\mathbf{k}} |\alpha_{\mathbf{k}}\rangle, \tag{2.1.8}$$

where

$$|\alpha_{\mathbf{k}}\rangle = \exp(\alpha_{\mathbf{k}} a_{\mathbf{k}}^\dagger - \alpha_{\mathbf{k}}^* a_{\mathbf{k}})\,|0\rangle_{\mathbf{k}}. \tag{2.1.9}$$

In the remainder of this chapter, we shall be mostly concerned with a single-mode coherent state. We shall therefore remove the index **k** from our definition in Eq. (2.1.9) and write

$$|\alpha\rangle = \exp(\alpha a^\dagger - \alpha^* a)\,|0\rangle. \tag{2.1.10}$$

In the following, we present alternative approaches to the coherent state.

2.2 The coherent state as an eigenstate of the annihilation operator and as a displaced harmonic oscillator state

Expression (2.1.10) was obtained by defining the coherent state of the radiation field $|\alpha\rangle$ as a state of the field which is generated by a classically oscillating current distribution. The same expression for $|\alpha\rangle$ can be obtained by defining it as an eigenstate of the annihilation operator a with an eigenvalue α, i.e.,

$$a|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.2.1}$$

An expression of $|\alpha\rangle$ in terms of the number state $|n\rangle$ is given by

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{2.2.2}$$

and since $|n\rangle = [(a^\dagger)^n/\sqrt{n!}]|0\rangle$ this can be written as

$$|\alpha\rangle = e^{\alpha a^\dagger}|0\rangle e^{-|\alpha|^2/2}. \tag{2.2.3}$$

Next we note that since $\exp(-\alpha^* a)|0\rangle = |0\rangle$, Eq. (2.2.3) can be rewritten as

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{2.2.4}$$

where

$$D(\alpha) = e^{-|\alpha|^2/2}e^{\alpha a^\dagger}e^{-\alpha^* a}. \tag{2.2.5}$$

Now, in view of the Baker–Hausdorff formula, if A and B are any two operators such that

$$[[A,B],A] = [[A,B],B] = 0, \tag{2.2.6}$$

then

$$e^{A+B} = e^{-[A,B]/2}e^A e^B. \tag{2.2.7}$$

If we write $A = \alpha a^\dagger, B = -\alpha^* a$, it follows that

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}, \tag{2.2.8}$$

in agreement with Eq. (2.1.10). Another equivalent antinormal form of $D(\alpha)$ is

$$D(\alpha) = e^{|\alpha|^2/2}e^{-\alpha^* a}e^{\alpha a^\dagger}. \tag{2.2.9}$$

The operator $D(\alpha)$ is a unitary operator, i.e.,

$$D^\dagger(\alpha) = D(-\alpha) = [D(\alpha)]^{-1}. \tag{2.2.10}$$

It acts as a displacement operator upon the amplitudes a and $a^\dagger$, i.e.,

$$D^{-1}(\alpha)aD(\alpha) = a + \alpha, \tag{2.2.11}$$

$$D^{-1}(\alpha)a^\dagger D(\alpha) = a^\dagger + \alpha^*. \tag{2.2.12}$$

The displacement property can be proved by writing

$$D^{-1}(\alpha)aD(\alpha) = e^{\alpha^* a}e^{-\alpha a^\dagger}ae^{\alpha a^\dagger}e^{-\alpha^* a}, \tag{2.2.13}$$

where we have used the form (2.2.9) for $D^{-1}(\alpha)$ and the form (2.2.5) for $D(\alpha)$. For any operators A and B

$$e^{-\alpha A}Be^{\alpha A} = B - \alpha[A, B] + \frac{\alpha^2}{2!}[A, [A, B]] + \ldots \tag{2.2.14}$$

For $A = a^\dagger, B = a$, this becomes

$$e^{-\alpha a^\dagger}ae^{\alpha a^\dagger} = a + \alpha. \tag{2.2.15}$$

Use of this result in Eq. (2.2.13) gives the displacement property (2.2.11) for $D(\alpha)$. The displacement property (2.2.12) can be proved in a similar way.

According to Eq. (2.2.4), a coherent state is obtained by applying the displacement operator on the vacuum state. The coherent state is therefore the displaced form of the harmonic oscillator ground state.

2.3 What is so coherent about coherent states?

To answer this question it is instructive to consider the coordinate representation of the oscillator number state $|n\rangle$. The coordinate representation of $|n\rangle$ is given by

$$\phi_n(q) = \langle q|n\rangle. \tag{2.3.1}$$

It follows from Eqs. (1.1.11) that

$$a = \frac{1}{\sqrt{2\hbar\nu}}\left(\nu q + \hbar\frac{\partial}{\partial q}\right), \quad a^\dagger = \frac{1}{\sqrt{2\hbar\nu}}\left(\nu q - \hbar\frac{\partial}{\partial q}\right), \tag{2.3.2}$$

where we have used $p = -i\hbar\partial/\partial q$. Equation (1.2.7) then leads to

$$\left(\nu q + \hbar\frac{\partial}{\partial q}\right)\phi_0(q) = 0. \tag{2.3.3}$$

A normalized solution of this equation is

$$\phi_0(q) = \left(\frac{\nu}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{\nu q^2}{2\hbar}\right). \tag{2.3.4}$$

Higher order eigenfunctions in the coordinate representation can be obtained from Eqs. (1.2.16), (2.3.1), and (2.3.2):

$$\begin{aligned}\phi_n(q) &= \frac{(a^\dagger)^n}{\sqrt{n!}}\phi_0(q) = \frac{1}{\sqrt{n!}}\frac{1}{(2\hbar\nu)^{n/2}}\left(\nu q - \hbar\frac{\partial}{\partial q}\right)^n \phi_0(q)\\ &= \frac{1}{(2^n n!)^{1/2}}H_n\left(\sqrt{\frac{\nu}{\hbar}}q\right)\phi_0(q),\end{aligned} \tag{2.3.5}$$

where H_n are the Hermite polynomials. These are the well-known eigenfunctions of the harmonic oscillator. It can be verified that these wave functions satisfy the orthonormality condition

$$\int_{-\infty}^{\infty}\phi_n^*(q)\phi_m(q)dq = \delta_{nm}. \tag{2.3.6}$$

It follows from the definition of the harmonic oscillator wave functions $\phi_n(q)$ that

$$\langle q\rangle = \int_{-\infty}^{\infty}\phi_n^*(q)q\phi_n(q)dq = 0. \tag{2.3.7}$$

Similarly

$$\langle p\rangle = 0, \tag{2.3.8}$$

$$\langle p^2\rangle = \hbar\nu\left(n+\frac{1}{2}\right), \tag{2.3.9}$$

$$\langle q^2\rangle = \frac{\hbar}{\nu}\left(n+\frac{1}{2}\right). \tag{2.3.10}$$

The uncertainties in the generalized momentum and coordinate variables are therefore given by

$$\begin{aligned}(\Delta p)^2 &= \langle p^2\rangle - \langle p\rangle^2\\ &= \hbar\nu\left(n+\frac{1}{2}\right),\end{aligned} \tag{2.3.11}$$

$$(\Delta q)^2 = \frac{\hbar}{\nu}\left(n+\frac{1}{2}\right). \tag{2.3.12}$$

The uncertainty product is

$$\Delta p\Delta q = \left(n+\frac{1}{2}\right)\hbar. \tag{2.3.13}$$

This has minimum possible value of $\hbar/2$ for the ground state wave function $\phi_0(q)$.

It is of special interest to find a wave packet which maintains the same variance Δq while undergoing simple harmonic motion. Such a wave function would correspond most closely to a classical field. In order to investigate this possibility we assume that, at time $t = 0$, the wave function $\psi(q,t)$ is of the form (2.3.4) of the minimum-uncertainty wave packet except that it is displaced in the positive q direction by an amount q_0. We then have

$$\psi(q,0) = \left(\frac{v}{\pi\hbar}\right)^{1/4} \exp\left[-\frac{v}{2\hbar}(q-q_0)^2\right]. \tag{2.3.14}$$

The time evolution of this wave packet is derived in Problem 2.3, where it is shown that the initial packet given by Eq. (2.3.14) implies that the probability density later in time is

$$|\psi(q,t)|^2 = \left(\frac{v}{\pi\hbar}\right)^{1/2} \exp\left[-\frac{v}{\hbar}(q-q_0\cos vt)^2\right]. \tag{2.3.15}$$

We note that the wave packet (2.3.14) oscillates back and forth in a simple harmonic oscillator potential without changing its shape, i.e., it sticks together or *coheres*. This is to be contrasted with the wave packet which is a delta function at $t = 0$, goes to a plane wave at $vt = \pi/2$, and is again a delta function at $vt = \pi$, see Section 2.5 for more details. Although the delta function packet returns to its original shape at the end of a period, it has a variance which is a strong function of time, i.e., it does not *cohere*.

The packet ψ has the minimum-uncertainty product allowed by quantum mechanics, namely $\Delta p \Delta q = \hbar/2$. These states therefore provide the closest quantum mechanical analog to a free classical single-mode field.

The minimum-uncertainty wave packet (2.3.14) which *coheres* in a simple harmonic oscillator potential is given by (Problem 2.4)

$$\psi(q,0) = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \langle q|n\rangle, \tag{2.3.16}$$

with $\alpha = (v/2\hbar)^{1/2} q_0$, where we use $\phi(q) = \langle q|n\rangle$. The state $|\alpha\rangle$ associated with $\psi(q,0)$ therefore has an expansion in number states identical to that for a coherent state, as given by Eq. (2.2.2). The minimum-uncertainty wave packet $\psi(q,0)$ is therefore the coordinate representation of the coherent state.

2.4 Some properties of coherent states

In this section, we list some important properties of the coherent states of the radiation field.

(a) The mean number of photons in the coherent state $|\alpha\rangle$ is given by

$$\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2. \tag{2.4.1}$$

The probability of finding n photons in $|\alpha\rangle$ is given by a Poisson distribution, i.e.,

$$p(n) = \langle n|\alpha\rangle\langle\alpha|n\rangle = \frac{|\alpha|^{2n}e^{-\langle n\rangle}}{n!} = \frac{\langle n\rangle^n e^{-\langle n\rangle}}{n!}. \tag{2.4.2}$$

where $\langle n\rangle = |\alpha|^2$. As we shall see in Chapter 11, the photon distribution for the laser approaches this distribution for sufficiently high excitations. In Fig. 2.2 we have plotted $p(n)$ versus n for different values of $|\alpha|^2$. It is seen that, for $|\alpha|^2 \leq 1$, $p(n)$ is maximum at $n = 0$, whereas, for $|\alpha|^2 > 1$, $p(n)$ has a peak at $n = |\alpha|^2$.

(b) As discussed earlier, the coherent state is a minimum-uncertainty state so that

$$\Delta p\Delta q = \frac{\hbar}{2}. \tag{2.4.3}$$

(c) The set of all coherent states $|\alpha\rangle$ is a complete set. To show this, we first consider the integral identity (with $\alpha = |\alpha|e^{i\theta}$)

$$\begin{aligned}\int(\alpha^*)^n\alpha^m e^{-|\alpha|^2}d^2\alpha &= \int_0^\infty |\alpha|^{n+m+1}e^{-|\alpha|^2}d|\alpha| \int_0^{2\pi} e^{i(m-n)\theta}d\theta \\ &= \pi n!\delta_{nm},\end{aligned} \tag{2.4.4}$$

in which the integration is carried out over the entire area of the complex plane. With the help of this identity it follows, on using the expansion (2.2.2) for the coherent states, that

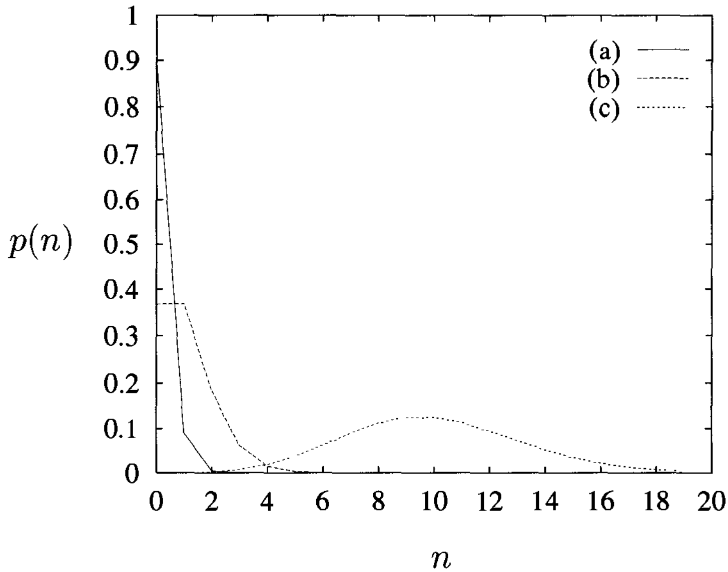
$$\int |\alpha\rangle\langle\alpha|d^2\alpha = \pi\sum_n |n\rangle\langle n|. \tag{2.4.5}$$

Since the Fock states $|n\rangle$ form a complete orthonormal set, the sum over n is simply the unit operator. We thus have

$$\frac{1}{\pi}\int |\alpha\rangle\langle\alpha|d^2\alpha = 1, \tag{2.4.6}$$

which is the completeness relation for the coherent states.

Fig. 2.2
The photon distribution $p(n)$ for a coherent state with (a) $|\alpha|^2 = 0.1$, (b) $|\alpha|^2 = 1$, and (c) $|\alpha|^2 = 10$.



(d) Two coherent states corresponding to different eigenstates α and α' are not orthogonal, i.e.,

$$\langle\alpha|\alpha'\rangle = \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha'\alpha^* - \frac{1}{2}|\alpha'|^2\right), \tag{2.4.7}$$

and

$$|\langle\alpha|\alpha'\rangle|^2 = \exp(-|\alpha - \alpha'|^2). \tag{2.4.8}$$

Here we see that, if the magnitude of $\alpha - \alpha'$ is much greater than unity, the states $|\alpha\rangle$ and $|\alpha'\rangle$ are nearly orthogonal to one another. The degree to which these wave functions overlap determines the size of the inner product $\langle\alpha|\alpha'\rangle$. A consequence of Eq. (2.4.7) is the fact that any coherent state can be expanded in terms of the other states:

$$\begin{aligned} |\alpha\rangle &= \frac{1}{\pi}\int d^2\alpha' |\alpha'\rangle\langle\alpha'|\alpha\rangle \\ &= \frac{1}{\pi}\int d^2\alpha' |\alpha'\rangle \exp\left(-\frac{1}{2}|\alpha|^2 + \alpha'^*\alpha - \frac{1}{2}|\alpha'|^2\right). \end{aligned} \tag{2.4.9}$$

This indicates that the coherent states are *overcomplete*.

2.5 Squeezed state physics

Natural philosophy, the union of experimental and theoretical science, abounds with wonderful examples of the fruitful interplay between experimental and theoretical thought. The 'ultraviolet catastrophe' observed in black-body radiation led Planck to introduce the notion of the quantum. These considerations led Einstein to the concept of 'stimulated emission' which was the key to understanding the differences between the radiation distributions of Planck and Wien. Stimulated emission is, of course, the basis for the laser which ushered in the modern era of quantum optics.

Squeezed states of the radiation field provide another, near term, example of the rich interplay between experiment and theory. By itself, the squeezing of states of the field is of limited interest. For example, the number state consisting of n photons clearly exists, but how to make it and who cares if we do?

One answer to the 'who cares?' question comes from the search for gravitational radiation. As is further discussed in Chapter 4, the acceleration of distant matter, e.g., the explosion of a supernova, leads to tiny forces on laboratory instruments. For example, an oscillating gravity wave can drive a mechanical oscillator which thus serves as a gravity wave detector.

But the amplitudes of oscillation generated by many sources of gravitational radiation are anticipated to be much smaller than the width of the ground state wave function. This prompted people to think about squeezing the ground state wave function (zero-point noise) of quantum mechanical oscillators.

That such 'squeezing' is possible in principle is made clear by considering the elementary quantum mechanics of the simple harmonic oscillator (SHO). As is depicted in Fig. 2.3, a wave packet which is sharply peaked (i.e., squeezed) initially will spread out and return to its initial state periodically. A little review of the SHO time evolution makes this clear.* Recall that the wave function at time t is related to that at $t = 0$ by the expression

$$\psi(x,t) = \int dx' G(x,x',t)\psi(x',0), \tag{2.5.1}$$

where the well known SHO propagator, as given in quantum mechanics texts, is

$$G(x,x',t) = \sqrt{\frac{m\nu}{2\pi\hbar|\sin\nu t|}} \exp\left\{\frac{im\nu}{2\hbar\sin\nu t}[(x^2+x'^2)\cos\nu t - 2xx']\right\}, \tag{2.5.2}$$

* See, for example, Sargent, Scully, and Lamb, *Laser Physics* [1974] Appendix H.

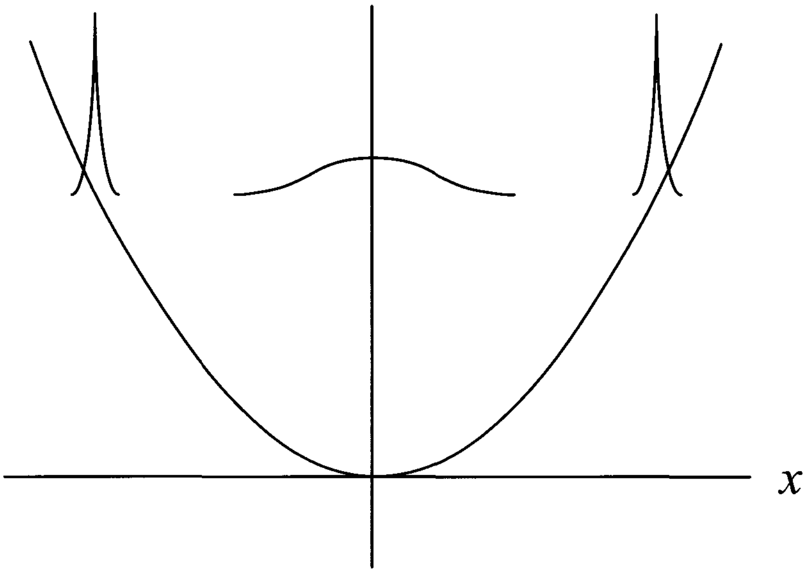


Fig. 2.3 Evolution of a squeezed state of a simple harmonic oscillator.

with m and v being the mass and frequency of the oscillator.

Now if we begin at $t = 0$ with a δ-function wave packet $\psi(x', 0) = \delta(x' - x_0)$ then at a time $t = \pi/2v$ later the wave function will be a plane wave; that is, our squeezed state evolves as

$$\psi(x, t = 0) = \delta(x - x_0), \tag{2.5.3a}$$

$$\psi(x, t = \pi/2v) = \sqrt{\frac{mv}{2\pi\hbar}} \exp\left[i\left(\frac{mvx_0}{\hbar}\right)x\right], \tag{2.5.3b}$$

$$\psi(x, t = \pi/v) = \delta(x + x_0). \tag{2.5.3c}$$

Thus, from Fig. 2.3 and Eqs. (2.5.3), we see that if we start with a sharp or squeezed state we will return to a sharp state every half period. In this sense we have the possibility of a kind of 'stroboscopic' measurement, in which we look at our oscillator at $t = 0,\ \pi/v,\ 2\pi/v, ...,$ so that we are not limited by the width of the ground state wave function.

Having motivated and illustrated squeezed states, let us proceed to a better understanding of these states by considering a *gedanken* experiment illustrating how we might prepare such states. To this end, let us return briefly to the question of how we might prepare a coherent state.

In classical mechanics we can excite a SHO into motion by, e.g., stretching the spring of Fig. 2.4 to a new equilibrium position and releasing it to produce oscillation. In quantum mechanics a similar procedure can be followed but we must be more specific about how

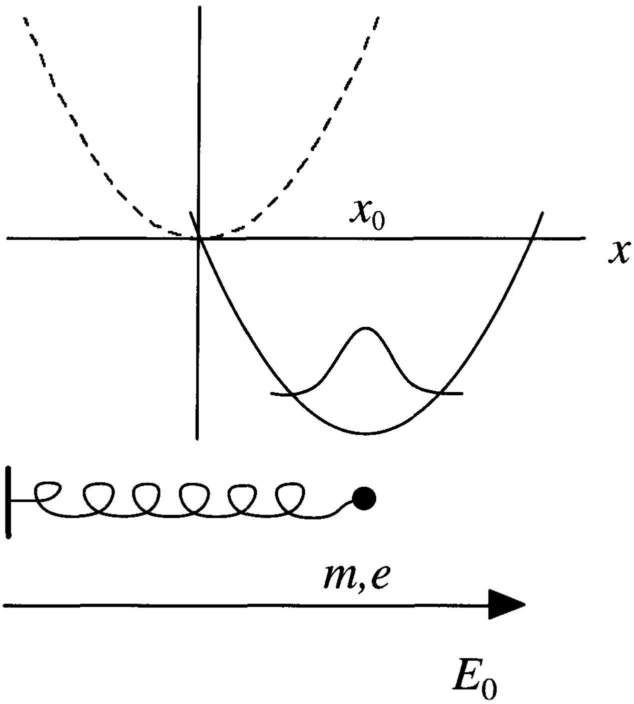


Fig. 2.4
Dashed potential applies for a spring-type SHO and causes a particle of mass m and charge e to oscillate about $x = 0$. Applying a dc electric field stretches the spring to a new equilibrium position x_0 about which the point charge particle now oscillates.

we prepare the initial state of the SHO. Let us envision a SHO characterized by mass m and charge e in a field E_0, as in Fig. 2.4; then the Hamiltonian is

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0x, \tag{2.5.4a}$$

which we may write as

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}k\left(x - \frac{eE_0}{k}\right)^2 - \frac{1}{2}k\left(\frac{eE_0}{k}\right)^2. \tag{2.5.4b}$$

We have in (2.5.4b) the well-known fact that applying a linear potential to a SHO just shifts its equilibrium point. Clearly the same solutions obtain. We have thus prepared a displaced ground state as in Fig. 2.4. And upon turning off the dc field, i.e., setting $E_0 = 0$, we will have a coherent state $|\alpha\rangle$ which oscillates without changing its shape.

It is to be noted that applying the dc field to the SHO is mathematically equivalent to applying the displacement operator (2.2.8) to the state $|0\rangle$. This is summarized in Fig. 2.4.

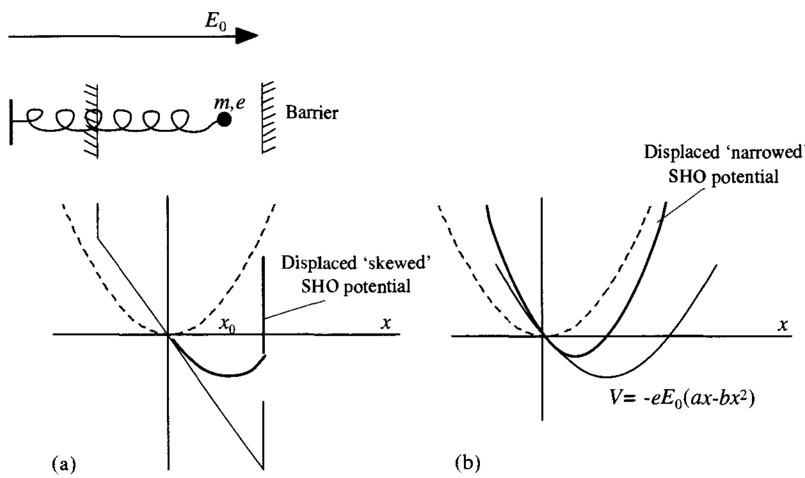


Fig. 2.5
(a) The SHO potential is first displaced by a dc elecric field and then 'skewed' by barriers which limit the charge oscillation to a finite region.
(b) The SHO potential is displaced and 'narrowed' by a quadratic displacement potential.

Next, let us consider how we might prepare a squeezed state. Suppose we again apply a dc field but this time with a 'wall' which limits the SHO to a finite region as in Fig. 2.5(a).

In such a case, it would be expected that the wave packet would be deformed or 'squeezed' when it is pushed against the barrier. Similarly the quadratic displacement potential of Fig. 2.5(b) would be expected to produce a squeezed wave packet. To see that this is indeed the case, consider the Hamiltonian for the SHO in the presence of the quadratic potential

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}kx^2 - eE_0(ax - bx^2), \tag{2.5.5a}$$

where the ax term will displace the oscillator and the bx^2 is added in order to give us a barrier to 'squeeze the packet against'. We rewrite (2.5.5a) as

$$\mathscr{H} = \frac{p^2}{2m} + \frac{1}{2}(k + 2ebE_0)x^2 - eaE_0x. \tag{2.5.5b}$$

From Eq. (2.5.5b) it is clear that we again have a displaced ground state, but this time with the larger effective spring constant $k' = k + 2ebE_0$. This, of course, means that we have a *squeezed* displaced wave packet as depicted in Fig. 2.6. This is the desired result.

In conclusion we note that, just as it is the creation operator part of the linear displacement potential which is most important in preparing a coherent state; we shall find that it is the two-photon $a^{\dagger 2}$ and a^2 contributions, contained within the bx^2 term in Eqs. (2.5.5), that are most important in preparing a squeezed coherent state.

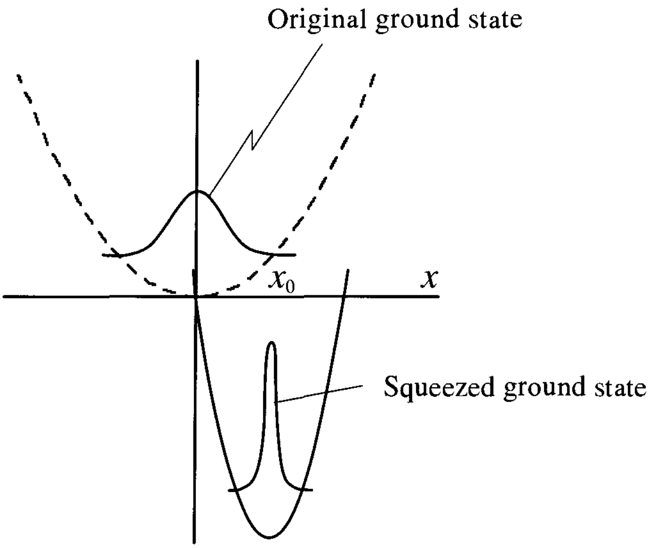


Fig. 2.6
The displaced 'narrowed' SHO potential squeezes the wave packet.

2.6 Squeezed states and the uncertainty relation

Having motivated the study and nature of squeezed states, let us consider what other properties we might expect from them. Consider two Hermitian operators A and B which satisfy the commutation relation

$$[A, B] = iC. \tag{2.6.1}$$

According to the Heisenberg uncertainty relation, the product of the uncertainties in determining the expectation values of two variables A and B is given by

$$\Delta A \Delta B \geq \frac{1}{2}|\langle C \rangle|. \tag{2.6.2}$$

A state of the system is called a squeezed state if the uncertainty in one of the observables (say A) satisfies the relation

$$(\Delta A)^2 < \frac{1}{2}|\langle C \rangle|. \tag{2.6.3}$$

If, in addition to the condition (2.6.3), the variances satisfy the minimum-uncertainty relation, i.e.,

$$\Delta A \Delta B = \frac{1}{2}|\langle C \rangle|, \tag{2.6.4}$$

then the state is called an ideal squeezed state.

In a squeezed state, therefore, the quantum fluctuations in one variable are reduced below their value in a symmetric minimum-uncertainty state ($(\Delta A)^2 = (\Delta B)^2 = |\langle C\rangle|/2$) at the expense of the corresponding increased fluctuations in the conjugate variable such that the uncertainty relation is not violated.

As an illustration, we consider a quantized single-mode electric field of frequency ν:

$$\mathbf{E}(t) = \mathscr{E}\hat{\epsilon}(ae^{-i\nu t} + a^\dagger e^{i\nu t}), \tag{2.6.5}$$

where a and $a^\dagger$ obey the commutation relation

$$[a, a^\dagger] = 1. \tag{2.6.6}$$

We introduce the Hermitian amplitude operators

$$X_1 = \frac{1}{2}(a + a^\dagger), \tag{2.6.7}$$

$$X_2 = \frac{1}{2i}(a - a^\dagger). \tag{2.6.8}$$

It is, of course, clear that X_1 and X_2 are essentially dimensionless position and momentum operators

$$x = \frac{\sqrt{2\hbar/m\nu}}{2}(a + a^\dagger),$$

$$p = \frac{\sqrt{2m\hbar\nu}}{2i}(a - a^\dagger).$$

It follows from the commutation relation (2.6.6) that X_1 and X_2 satisfy

$$[X_1, X_2] = \frac{i}{2}. \tag{2.6.9}$$

In terms of these operators, Eq. (2.6.5) can be rewritten as

$$\mathbf{E}(t) = 2\mathscr{E}\hat{\epsilon}(X_1 \cos\nu t + X_2 \sin\nu t). \tag{2.6.10}$$

The Hermitian operators X_1 and X_2 are now readily seen to be the amplitudes of the two quadratures of the field having a phase difference $\pi/2$. From Eq. (2.6.9), the uncertainty relation for the two amplitudes is

$$\Delta X_1 \Delta X_2 \geq \frac{1}{4}. \tag{2.6.11}$$

A squeezed state of the radiation field is obtained if

$$(\Delta X_i)^2 < \frac{1}{4} \qquad (i = 1 \text{ or } 2). \tag{2.6.12}$$

An ideal squeezed state is obtained if in addition to Eq. (2.6.12), the relation

$$\Delta X_1 \Delta X_2 = \frac{1}{4} \tag{2.6.13}$$

also holds.

In the next section we will consider the two-photon coherent state which is an example of an ideal squeezed state. Here we mention that the coherent state $|\alpha\rangle$ and the Fock state $|n\rangle$ are not squeezed states. It follows from Eq. (2.6.7) that, in a coherent state,

$$\begin{aligned}(\Delta X_1)^2 &= \langle\alpha|X_1^2|\alpha\rangle - (\langle\alpha|X_1|\alpha\rangle)^2 \\ &= \frac{1}{4}\langle\alpha|[a^2 + aa^\dagger + a^\dagger a + (a^\dagger)^2]|\alpha\rangle - \frac{1}{4}[\langle\alpha|(a + a^\dagger)|\alpha\rangle]^2 \\ &= \frac{1}{4}.\end{aligned} \tag{2.6.14}$$

Similarly

$$(\Delta X_2)^2 = \frac{1}{4}. \tag{2.6.15}$$

In a similar manner, in a Fock state,

$$\begin{aligned}(\Delta X_1)^2 &= \langle n|X_1^2|n\rangle - (\langle n|X_1|n\rangle)^2, \\ &= \frac{1}{4}(2n+1),\end{aligned} \tag{2.6.16}$$

$$(\Delta X_2)^2 = \frac{1}{4}(2n+1). \tag{2.6.17}$$

In Fig. 2.7 error contours of the uncertainties in X_1 and X_2, along with the corresponding graphs of the electric field versus time are shown for a coherent state, a squeezed state with reduced noise in X_1, and a squeezed state with reduced noise in X_2. Each point in the error contour for various states corresponds to a wave with a certain amplitude and a certain phase. A summation of all such waves in the error contours thus leads to the uncertainties of the electric field represented by the shaded region. A coherent state (Fig. 2.7(a)), having identical uncertainties in both X_1 and X_2, has a constant value for the variance of the electric field. A squeezed state with reduced noise in X_1 (Fig. 2.7(b)) has reduced uncertainty in the amplitude at the expense of large uncertainty in the phase of the electric field whereas the situation is reversed for a squeezed state with reduced noise in X_2 (Fig. 2.7(c)).

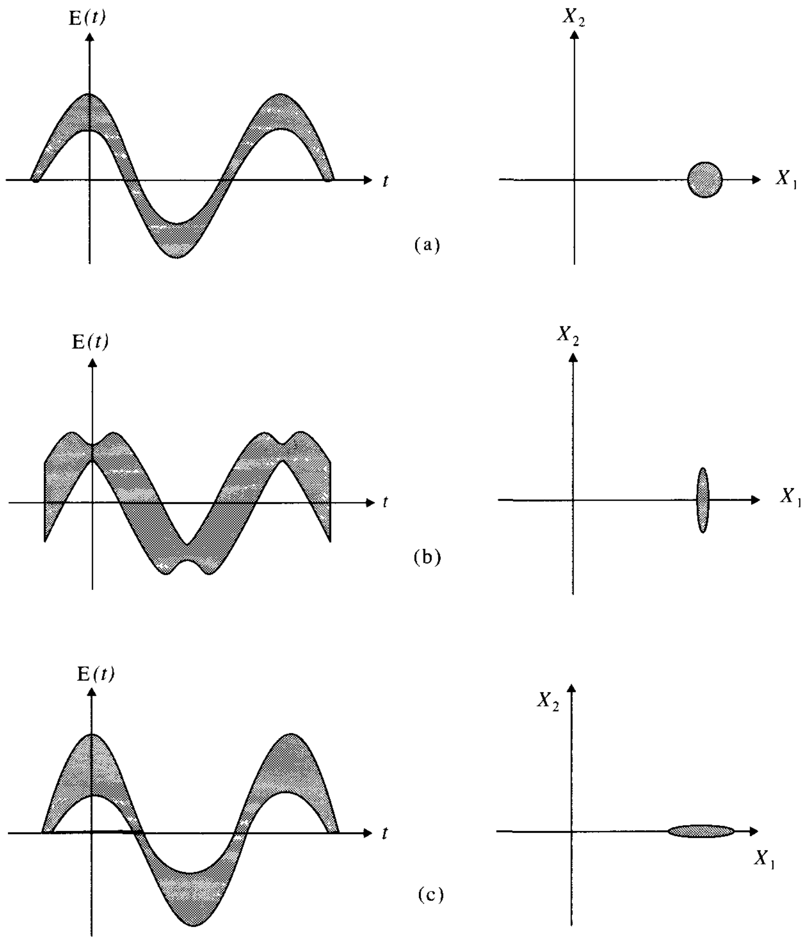


Fig. 2.7
Error contours and the corresponding graphs of electric field versus time for (a) a coherent state, (b) a squeezed state with reduced nosie in X_1, and (c) a squeezed state with reduced noise in X_2. (From C. Caves, *Phys. Rev. D* **23**, 1693 (1981).)

2.7 The squeeze operator and the squeezed coherent states

In Section 2.5 we found that quadratic terms in x, i.e., terms of the form $(a + a^\dagger)^2$, were important in the preparation of squeezed states. With that thought in mind, we are naturally motivated to consider degenerate parametric processes in connection with the generation of such states of the radiation field. In fact, much of squeezed state physics is nicely illustrated by the degenerate parametric process, as discussed in Chapter 16. The associated two-photon Hamiltonian can be written as

$$\mathscr{H} = i\hbar \left(g a^{\dagger 2} - g^* a^2\right), \tag{2.7.1}$$

where g is a coupling constant. Hence the state of the field generated by this expression is

$$|\psi(t)\rangle = e^{(ga^{\dagger 2}-g^{*}a^{2})t}\,|0\rangle \tag{2.7.2}$$

and this leads us to define the unitary squeeze operator

$$S(\xi) = \exp\left(\frac{1}{2}\xi^{*}a^{2} - \frac{1}{2}\xi a^{\dagger 2}\right), \tag{2.7.3}$$

where $\xi = r\exp(i\theta)$ is an arbitrary complex number. It is easy to see that

$$S^{\dagger}(\xi) = S^{-1}(\xi) = S(-\xi). \tag{2.7.4}$$

A straightforward application of the formula

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \ldots, \tag{2.7.5}$$

leads to the following useful unitary transformation properties of the squeeze operator

$$S^{\dagger}(\xi)aS(\xi) = a\cosh r - a^{\dagger}e^{i\theta}\sinh r, \tag{2.7.6}$$

$$S^{\dagger}(\xi)a^{\dagger}S(\xi) = a^{\dagger}\cosh r - ae^{-i\theta}\sinh r. \tag{2.7.7}$$

If we define a rotated complex amplitude at an angle $\theta/2$

$$Y_1 + iY_2 = (X_1 + iX_2)e^{-i\theta/2}, \tag{2.7.8}$$

it follows from Eq. (2.7.6) that

$$S^{\dagger}(\xi)(Y_1 + iY_2)S(\xi) = Y_1e^{-r} + iY_2e^{r}. \tag{2.7.9}$$

A squeezed coherent state $|\alpha,\xi\rangle$ is obtained by first acting with the displacement operator $D(\alpha)$ on the vacuum followed by the squeeze operator $S(\xi)$, i.e.,

$$|\alpha,\xi\rangle = S(\xi)D(\alpha)|0\rangle, \tag{2.7.10}$$

with $\alpha = |\alpha|\exp(i\varphi)$. As discussed earlier, whereas a coherent state is generated by linear terms in a and $a^{\dagger}$ in the exponent, the squeezed coherent state requires quadratic terms.

In the following we discuss some properties of the squeezed coherent state since it is a canonical example of a squeezed state.

2.7.1 Quadrature variance

The operator expectation values of the state $|\alpha,\xi\rangle$ can be determined from the definition (2.7.10) by making use of the transformation properties of the displacement and squeezing operators (Eq. (2.7.3)). It then follows that

$$\begin{aligned}\langle a\rangle &= \langle\alpha,\xi|a|\alpha,\xi\rangle\\ &= \langle 0|D^\dagger(\alpha)S^\dagger(\xi)aS(\xi)D(\alpha)|0\rangle\\ &= \langle\alpha|(a\cosh r - a^\dagger e^{i\theta}\sinh r)|\alpha\rangle\\ &= \alpha\cosh r - \alpha^* e^{i\theta}\sinh r, \qquad (2.7.11)\\ \langle a^2\rangle &= \langle(a^\dagger)^2\rangle^*\\ &= \langle 0|D^\dagger(\alpha)S^\dagger(\xi)a^2S(\xi)D(\alpha)|0\rangle\\ &= \langle\alpha|S^\dagger(\xi)aS(\xi)S^\dagger(\xi)aS(\xi)|\alpha\rangle\\ &= \alpha^2\cosh^2 r + (\alpha^*)^2e^{2i\theta}\sinh^2 r - 2|\alpha|^2e^{i\theta}\sinh r\cosh r\\ &\quad -e^{i\theta}\cosh r\sinh r, \qquad (2.7.12)\\ \langle a^\dagger a\rangle &= |\alpha|^2(\cosh^2 r + \sinh^2 r) - (\alpha^*)^2e^{i\theta}\sinh r\cosh r\\ &\quad -\alpha^2e^{-i\theta}\sinh r\cosh r + \sinh^2 r. \qquad (2.7.13)\end{aligned}$$

The variances of the rotated amplitudes Y_1 and Y_2 can be determined from these expectation values. On substituting for X_1 and X_2 from Eqs. (2.6.7) and (2.6.8) into Eq. (2.7.8) we obtain

$$Y_1 + iY_2 = a\exp(-i\theta/2), \qquad (2.7.14)$$

so that

$$\begin{aligned}(\Delta Y_1)^2 &= \langle Y_1^2\rangle - \langle Y_1\rangle^2\\ &= \frac{1}{4}\langle(ae^{-i\theta/2} + a^\dagger e^{i\theta/2})^2\rangle - \frac{1}{4}(\langle ae^{-i\theta/2} + a^\dagger e^{i\theta/2}\rangle)^2\\ &= \frac{1}{4}\langle a^2e^{-i\theta} + a^{\dagger 2}e^{i\theta} + aa^\dagger + a^\dagger a\rangle\\ &\quad -\frac{1}{4}(\langle ae^{-i\theta/2} + a^\dagger e^{i\theta/2}\rangle)^2 = \frac{1}{4}e^{-2r}, \qquad (2.7.15)\\ (\Delta Y_2)^2 &= \frac{1}{4}e^{2r}, \qquad (2.7.16)\\ \Delta Y_1\Delta Y_2 &= \frac{1}{4}. \qquad (2.7.17)\end{aligned}$$

A squeezed coherent state is therefore an ideal squeezed state. As shown in Fig. 2.8, in the complex amplitude plane the coherent state error circle is *squeezed* into an *error ellipse* of the same area. The principal axes of the ellipse lie along Y_1 and Y_2 rotated at an angle $\theta/2$ from X_1 and X_2, respectively. The degree of squeezing is determined by $r = |\xi|$ which is therefore called the squeeze parameter.

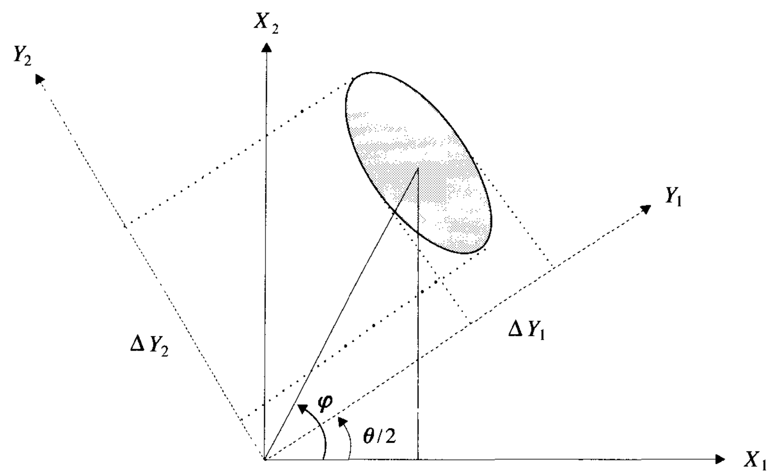


Fig. 2.8
Error contour for a squeezed coherent state.

2.8 Multi-mode squeezing

The single-mode two-photon coherent state can be generalized to a multi-mode squeezed state by using a generator which incorporates the product of annihilation (and creation) operators for correlated pairs of modes symmetrically placed around a mode of frequency, say, ν. First, we discuss the simple case of two-mode squeezing and then generalize it to the multi-mode case. The two-mode squeezed state is obtained by the action of the unitary operator

$$S(\xi) = e^{\xi^* a_{\nu+\nu'} a_{\nu-\nu'} - \xi a^\dagger_{\nu+\nu'} a^\dagger_{\nu-\nu'}}, \tag{2.8.1}$$

on the two-mode vacuum.

To show that the operators spanning the two modes exhibit squeezing, we define collective creation and destruction operators

$$b^\dagger = \frac{1}{\sqrt{2}}\left[a^\dagger_{\nu+\nu'} + e^{i\delta} a^\dagger_{\nu-\nu'}\right], \tag{2.8.2}$$

$$b = \frac{1}{\sqrt{2}}\left[a_{\nu+\nu'} + e^{-i\delta} a_{\nu-\nu'}\right]. \tag{2.8.3}$$

The in-phase and in-quadrature components are given by

$$b_1 = \frac{1}{2}(b + b^\dagger), \tag{2.8.4}$$

$$b_2 = \frac{1}{2i}(b - b^\dagger). \tag{2.8.5}$$

The corresponding uncertainty relation is

$$\Delta b_1 \Delta b_2 \geq \frac{1}{4}. \tag{2.8.6}$$

The variances in the two components in the two-mode squeezed vacuum are

$$(\Delta b_1)^2 = \frac{1}{4}\left[\exp(-2r)\cos^2\left(\frac{\delta}{2}-\frac{\theta}{2}\right)+\exp(2r)\sin^2\left(\frac{\delta}{2}-\frac{\theta}{2}\right)\right], \tag{2.8.7}$$

$$(\Delta b_2)^2 = \frac{1}{4}\left[\exp(2r)\cos^2\left(\frac{\delta}{2}-\frac{\theta}{2}\right)+\exp(-2r)\sin^2\left(\frac{\delta}{2}-\frac{\theta}{2}\right)\right]. \tag{2.8.8}$$

For the particular choices of the phase $\delta - \theta = 0$ and π, it is an ideal squeezed state with reduced fluctuations in b_1 and b_2, respectively.

In a similar manner, a large number of modes of the vacuum can be squeezed. The multi-mode squeeze operator is defined as

$$S[\xi(\nu)] = \int \frac{d\nu'}{2\pi} \exp\left[\xi^*(\nu')a_{\nu+\nu'}a_{\nu-\nu'} - \xi(\nu')a^\dagger_{\nu+\nu'}a^\dagger_{\nu-\nu'}\right]. \tag{2.8.9}$$

Here the integration is over the positive half-band of frequencies and $\xi(\nu) = r(\nu)\exp[i\theta(\nu)]$. A multi-mode squeezed coherent state is obtained, as in definition (2.7.10), by first displacing the vacuum and then squeezing it through a multi-mode displacement operator

$$|\alpha(\nu), \xi(\nu)\rangle \equiv S[\xi(\nu)]D[\alpha(\nu)]|\widetilde{0}\rangle, \tag{2.8.10}$$

where $|\widetilde{0}\rangle$ is a multi-mode vacuum state.

Problems

2.1 Show that

$$a^\dagger|\alpha\rangle\langle\alpha| = \left(\alpha^* + \frac{\partial}{\partial\alpha}\right)|\alpha\rangle\langle\alpha|,$$

and

$$|\alpha\rangle\langle\alpha|a = \left(\alpha + \frac{\partial}{\partial\alpha^*}\right)|\alpha\rangle\langle\alpha|.$$

2.2 Show that the expectation value of the displacement operator $D(\alpha)$ for a thermal field is given by

$$\langle D(\alpha)\rangle = \exp\left[-|\alpha|^2\left(\langle n\rangle + \frac{1}{2}\right)\right],$$

where $\langle n\rangle$ is the mean number of photons in the field.

2.3 The time evolution of the wave packet (2.3.14) is determined by the Schrödinger equation for the harmonic oscillator

$$i\hbar\frac{\partial\psi}{\partial t} = \left(-\frac{\hbar^2}{2}\frac{\partial^2}{\partial q^2} + \frac{\nu^2 q^2}{2}\right)\psi.$$

A general solution of this equation can be given in terms of the stationary wave functions

$$\psi(q,t) = \sum_{n=0}^{\infty} a_n \phi_n(q) e^{-iE_n t/\hbar},$$

where $E_n = (n+1/2)\hbar\nu$ and a_n are arbitrary coefficients. Using the orthonormality conditions on the wave functions $\phi_n(q)$, find a_n and hence prove Eq. (2.3.15).

2.4 Derive Eq. (2.3.16).

2.5 An alternate definition of a squeezed coherent state is

$$|\alpha, \xi\rangle = D(\alpha)S(\xi)|0\rangle,$$

where $\xi = r\exp(i\theta)$. Show that the variances in the quadrature components Y_1 and Y_2, such that

$$Y_1 + iY_2 = ae^{-i\theta/2},$$

are given by

$$(\Delta Y_1)^2 = \frac{1}{4}e^{-2r},$$

$$(\Delta Y_2)^2 = \frac{1}{4}e^{2r}.$$

2.6 Consider a two-mode squeezed state defined by

$$|\alpha_1, \alpha_2, \xi\rangle = D_1(\alpha_1)D_2(\alpha_2)S_{12}(\xi)|0\rangle,$$

where

$$D_i(\alpha_i) = \exp(\alpha_i a_i^\dagger - \alpha_i^* a_i) \quad (i = 1, 2),$$

is the coherent displacement operator for the two modes described by destruction and creation operators a_i and $a_i^\dagger$, respectively,

$$S_{12}(\xi) = \exp(\xi^* a_1 a_2 + \xi a_1^\dagger a_2^\dagger)$$

is the two-mode squeeze operator, and $|0\rangle$ is the two-mode vacuum state. Show that there is no squeezing in the two individual modes. (Hint: see S. M. Barnett and P. L. Knight, *J. Opt. Soc. Am. B* **2**, 467 (1985).)

2.7 A state is said to be squeezed in the Nth order if $\langle(\Delta X_i)^N\rangle$ ($i = 1$ or 2) is lower than its corresponding coherent state value. Here

$$X_1 = \frac{1}{2}(a + a^\dagger),$$
$$X_2 = \frac{1}{2i}(a - a^\dagger).$$

Show that the condition of the Nth-order squeezing is

$$q^N < 0,$$

where

$$q^N = (\Delta X_i)^N - \left(\frac{1}{4}\right)^{N-2} (N-1)!!$$

(Hint: see C. K. Hong and L. Mandel, *Phys. Rev. Lett.* **54**, 323 (1985).)

2.8 Consider the Hermitian operators corresponding to the real and imaginary parts of the square of the complex amplitude of the field

$$X_1 = \frac{1}{2}(a^2 + a^{\dagger 2}),$$
$$X_2 = \frac{1}{2i}(a^2 - a^{\dagger 2}).$$

Show that the squeezing condition is

$$\langle \Delta X_i^2 \rangle < \langle a^\dagger a \rangle + \frac{1}{2} \quad (i = 1 \text{ or } 2).$$

This type of squeezing is called amplitude-squared squeezing. Show that the amplitude-squared squeezing is a nonclassical effect. (Hint: see M. Hillery, *Phys. Rev. A* **36**, 3796 (1987).)

References and bibliography

Coherent states of the radiation field

E. Schrödinger, *Naturwissenschaften* **14**, 664 (1926).
E. H. Kennard, *Z. Physik* **44**, 326 (1927).
R. J. Glauber, *Phys. Rev.* **130**, 2529 (1963).
R. J. Glauber, in *Quantum Optics and Electronics*, Les Houches, ed. C. DeWitt, A. Blandin, and C. Cohen-Tannoudji (Gordon and Breach, New York 1965).
L. Mandel and E. Wolf, *Rev. Mod. Phys.* **37**, 231 (1965).
J. R. Klauder and B. S. Skagerstam, *Coherent States*, (World Scientific, Singapore 1985). This book contains an excellent collection of papers on coherent states.

Review of squeezed states

D. F. Walls, *Nature* **306**, 141 (1983).
G. Leuchs, in *Non-equilibrium Quantum Statistical Physics*, ed. G. T. Moore and M. O. Scully (Plenum, New York 1985), p. 329.
R. Loudon and P. L. Knight, *J. Mod. Opt.* **34**, 709 (1987).
K. Zaheer and M. S. Zubairy, in *Advances in Atomic, Molecular and Optical Physics* Vol. 28, ed. D. R. Bates and B. Bederson (Academic Press, New York 1990), p. 143.

Squeezed coherent state

D. Stoler, *Phys. Rev. D* **1**, 3217 (1970).
C. Y. E. Lu, *Lett. Nuovo Cim.* **2**, 1241 (1971).
M. Sargent III, M. O. Scully, and W. E. Lamb, Jr., *Laser Physics*, (Addison-Wesley, Mass. 1974).
H. P. Yuen, *Phys. Rev. A* **13**, 2226 (1976).
J. N. Hollenhorst, *Phys. Rev. D* **19**, 1669 (1979).
W. Schleich and J. A. Wheeler, *Nature* **326**, 574 (1987); *J. Opt. Soc. Am. B* **4**, 1715 (1987).

Applications of squeezed states to interferometry

C. M. Caves, *Phys. Rev. D* **23**, 1693 (1981).
M. Xiao, L. A. Wu, and H. J. Kimble, *Phys. Rev. Lett.* **59**, 278 (1987).
J. Gea-Banacloche and G. Leuchs *J. Opt. Soc. Am. B* **4**, 1667 (1987).

Multi-mode squeezing

C. M. Caves, *Phys. Rev. D* **26**, 1817 (1982).
S. M. Barnett and P. L. Knight, *J. Opt. Soc. Am. B* **2**, 467 (1985).
C. M. Caves and B. L. Schumaker, *Phys. Rev. A* **31**, 3068 (1985).
B. L. Schumaker and C. M. Caves, *Phys. Rev. A* **31**, 3093 (1985).

Higher order squeezing

R. A. Fisher, M. M. Nieto, and V. D. Sandberg, *Phys. Rev. D* **29**, 1107 (1984).
M. Hillery, M. S. Zubairy, and K. Wodkiewicz, *Phys. Lett.* **103A**, 259 (1984).
G. D'Ariano, M. Rasetti, and N. Vadacchino, *Phys. Rev. D* **32**, 1034 (1985).
C. K. Hong and L. Mandel, *Phys. Rev. Lett.* **54**, 323 (1985).
S. L. Braunstein and R. I. McLachlam, *Phys. Rev. A* **35**, 1659 (1987).
M. Hillery, *Phys. Rev. A* **36**, 3796 (1987); *Opt. Commun.* **62**, 135 (1987).